Journal of Statistical Physics, Vol. 72, Nos. 1/2, 1993

A Functional Central Limit Theorem for a Nonequilibrium Model of Interacting Particles with Unbounded Intensity

Paul Hubert Bezandry,[1] Xavier Fernique,[1] and Gaston Giroux[2]

Received September 22, 1992

Under suitable physically reasonable initial assumptions, a functional central limit theorem is obtained for a nonequilibrium model of randomly interacting particles with unbounded jump intensity. This model is related to a nonlinear Boltzmann-type equation.

KEY WORDS: Boltzmann equation; fluctuation; unbounded intensity; nonlinear evolution equation; Markov process; weak convergence.

1. INTRODUCTION

Problems of nonequilibrium statistical physics stimulate new developments in the theory of stochastic processes, which in turn provides a mathematically rigourous foundation for this subject. A valuable goal is thus the construction of models adapted to the study of nonequilibrium properties of various physical systems of subsystems. For some of these, the subsystems (also called particles) engage in interactions with the following two properties: they are binary and the result of an interaction is random. The Boltzmann model is such an example. On the other hand, most of the results in the stochastic processes literature are for the spatially homogeneous case. From the physical viewpoint, spatially homogeneous systems of such interacting particles are meaningless. However, in his book, Skoróhod[(12)] has shown that these artificial systems approximate sufficiently well mollified models of the Boltzmann type, at least to obtain the existence of

[1] Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 67084 Strasbourg Cedex, France.
[2] Université de Sherbrooke, Sherbrooke J1K 2R1, Québec, Canada.

0022-4715/93/0700-0329$07.00/0

the solution of the non-space-homogeneous mollified nonlinear Boltzmann-like evolution equation that he studied.

Related to Skorohod's approach is the problem of the fluctuation of the empirical laws around the solution. It seems reasonable, as a first step, to study these fluctuations for some space-homogeneous cases.

Already Kac[8] considered the fluctuation problem for his caricature of a space-homogeneous Maxwellian gas. He regarded the problem as a central limit theorem in which the limit process yields an infinite-dimensional Ornstein–Uhlenbeck process. These situations were clarified by McKean,[10] but the discussion lacks rigor except for his two-velocities model. The difficulty arises from the infinite-dimensional feature of the analysis employed. Later, Tanaka,[15] for the equilibrium case, and Uchiyama,[16] for the nonequilibrium case, succeeded in giving rigorous proofs for Kac's caricature. They used general results in infinite dimensions due to Ito.[6] Their limit theorems are stated as results for processes with values in the space of tempered distributions. The equilibrium case for hard spheres with a cutoff was also treated by Uchiyama[17] using the same space. Later, Ferland *et al.*[3] started a new proof for the nonequilibrium case of Kac's caricature, which can be extended to other models (see Ferland and Roberge[4]). The processes are in a suitable Hilbert space equipped with his weak topology. They used general results due to Fernique.[5]

All the models mentioned above are of bounded intensity. In this paper, we discuss a space-homogeneous model akin to the origin hard-sphere example of Boltzmann. It is a model with an unbounded intensity and we have to modify the approach in ref. 3. Our one-dimensional reduction of the hard-sphere model has regular properties that allow a change of variables which leads to expression (8) below. We are then able to write the fluctuations more explicitly as elements of a well-chosen Hilbert space. With this representation, we can then show the convergence of the fluctuation processes in another Hilbert space. It is a functional central limit theorem. Such functional results are needed to control the error done by a simulation (see Wagner,[18] where a uniform control is already needed to show the convergence of the empirical laws).

A stepping stone to our study is ref. 14; this is because we have not tried to adapt Skorohod's approach to our case. We are planning to do that in the near future, since, with this methodology, the existence of a solution of the nonlinear evolution equation is a consequence of the convergence of the empirical laws. For more information about nonequilibrium processes, the reader may consult Keizer's[9] and Spohn's[13] books.

2. THE SETUP

We study in this paper a finite system of particles which is characterized by an intensity function for the jumps and by a binary interacting kernel. It is a spatially homogeneous model with unbounded intensity, which is a one-dimensional analog of what is known as the Boltzmann hard-sphere model. This is one of the models introduced in ref. 14. According to this work (see Theorem 2.1 below), for each $n \geqslant 2$ there exists a unique probability law P^n on $D([0, T]; \mathfrak{R}^n_+)$ such that:

1. $P^n \circ (X^n_0)^{-1} = u^n$.

2. For at least the $f \in bB$ (the real-valued bounded Borel functions on $\mathfrak{R}_+$)

$$f(X^n_t) - f(X^n_0) - \frac{1}{n} \sum_{i<j} \int_0^t (Af^{i,j})(X^n_s)\, ds \tag{1}$$

is a martingale, where

$$(Af^{i,j})(X^n_s) = \int_0^1 [f(X^n_s + \Delta_{i,j}(r)) - f(X^n_s)]\, r\, |X^n_{i,s} - X^n_{j,s}|\, dr$$

$$\Delta_{i,j}(r) = h(X^n_{j,s}, X^n_{i,s}; r)e_i + h(X^n_{i,s}, X^n_{j,s}; r)e_j$$

$$h(x, y; r) = r(y - x)$$

X^n is a Markov process which behaves in the following way: if, at some time, the process is at $(x_1,..., x_n)$, it waits an exponential time with parameter

$$\frac{1}{2n} \sum_{i<j} |x_i - x_j|$$

and then changes its state to

$$(x_1,..., (1-r)x_i + rx_j,..., rx_i + (1-r)x_j,..., x_n)$$

with probability

$$\frac{2|x_i - x_j|\, r\, dr}{\sum_{i<j} |x_i - x_j|}$$

To this sequence of processes is associated a nonlinear Boltzmann-type equation:

$$\langle u_t, \varphi \rangle - \langle u_0, \varphi \rangle = \tfrac{1}{2} \int_0^t \langle u_s, A(u_s)\varphi \rangle\, ds$$

where u_0 is a given probability measure on $(\mathfrak{R}_+, \mathscr{B}(\mathfrak{R}_+))$ and, for $\mu \in \mathscr{M}_1^+(\mathfrak{R}_+)$,

$$(A(\mu)\varphi)(x) = \langle \mu, (A_1\varphi)(x, \cdot)\rangle$$

with

$$(A_k\varphi)(x, y) = \int_0^1 \{(\varepsilon^r\varphi)(x, y)\}^k r\,|x-y|\,dr$$

and

$$(\varepsilon^r\varphi)(x, y) = \varphi((1-r)x + ry) + \varphi(rx + (1-r)\,y) - \varphi(x) - \varphi(y)$$

Now let

$$\alpha_t^n = (1/n) \sum_{i=1}^{n} \delta_{X_{i,t}^n}$$

and let $\hat{P}^n$ be the probability measure on $D([0, T], \mathscr{M}_1^+(\mathfrak{R}_+))$ which is the law of

$$t \mapsto \alpha_t^n$$

the nth empirical process. The following results have been shown in ref. 14.

Theorem 2.1.

1. If $E[\langle \alpha_0^n, x^2\rangle] < \infty$, then there exists a unique P^n such that properties 1 and 2 above are true.
2. If $\langle u_0, x^2\rangle < \infty$, then the Boltzmann-like equation has a unique solution which satisfies

$$\sup_{0 \leqslant s \leqslant t} \langle u_s, x^2\rangle < \infty$$

3. If α_0^n weakly converges to u_0 and:

   (a) $\forall n$, u^n is exchangeable

   (b) $\exists a$ such that $\forall n$, $\langle \alpha_0^n, x\rangle \leqslant a$ a.e.

   (c) $\sup_n E[\langle \alpha_0^n, x^2\rangle] < \infty$

   (d) $\langle u_0, x^2\rangle < \infty$

   then $\hat{P}^n \Rightarrow \delta_{\{t \mapsto u_t\}}$; moreover, we have

$$\sup_n \sup_{0 \leqslant s \leqslant t} E[\langle \alpha_s^n, x^2\rangle] < \infty$$

In particular, for each $\varphi \in C_b(R_+)$ and $\beta \geq 1$, we have

$$E[|\langle \alpha_s^n - u_s, \varphi \rangle|^\beta] \to 0 \tag{2}$$

But to show the uniqueness of the limit fluctuation process we will need the following extension.

Lemma 2.1. Assume α_0^n weakly converges to u_0.

1. Under **(a)–(d)**, for each $\varphi \in C(R_+)$ such that $|\varphi(x)| \leq C(1+x)$, we have

$$E[|\langle \alpha_s^n - u_s, \varphi \rangle|] \to 0$$

2. If in assumptions **(c)** and **(d)**, x^2 is replaced by x^4, then for each $\psi \in C(R_+ \times R_+)$ such that $|\psi(x, y)| \leq C(1 + x + y)$, we have

$$E[|\langle (\alpha_s^n - u_s) \otimes (\alpha_s^n - u_s), \psi \rangle|] \to 0$$

Proof. 1. Using a continuous function $0 \leq g \leq 1$ equal to zero on $[M + \delta, \infty[$ and to one on $[0, M]$, we have, by (2), $E[|\langle (\alpha_s^n - u_s), g\varphi \rangle|] \to 0$. So, for $M > 1$, we get

$$\begin{aligned}
\limsup E[|\langle (\alpha_s^n - u_s), \varphi \rangle|] &\leq \limsup E[\langle (\alpha_s^n + u_s), (1-g)|\varphi| \rangle] \\
&\leq \limsup \frac{C}{M} E[\langle (\alpha_s^n + u_s), x^2 \rangle] \\
&\leq \frac{C}{M} \{\langle u_s, x^2 \rangle + \sup_n E[\langle \alpha_s^n, x^2 \rangle]\}
\end{aligned}$$

Theorem 2.1 gives the first part of the lemma.

2. It has been shown in ref. 14 that if for $r \geq 2$ we have

$$\sup_n E[\langle \alpha_0^n, |x|^r \rangle] < \infty$$

then we get

$$\sup_n \sup_{0 \leq s \leq t} E[(\alpha_s^n, |x|^r \rangle] < \infty$$

with a similar statement for u_s. Now, by Stone–Weierstrass, we are able to approach ψ on a compact $[0, M] \times [0, M]$ by a function of the form

$$\sum_{k=1}^{p} \varphi_k(x) \, \psi_k(y); \qquad \varphi_k, \psi_k \in C_b$$

That is to say, for $M>0$ and $\varepsilon \in]0, 1/2]$, there exist $p \in \mathcal{N}$ and $\varphi_k, \psi_k \in C_b$, $1 \leqslant k \leqslant p$, such that

$$\sup_{x \leqslant M, y \leqslant M} \left| \psi(x, y) - \sum_{k=1}^{p} \varphi_k(x)\, \psi_k(y) \right| \leqslant \varepsilon/(16M^2) \tag{3}$$

and

$$\sup_{\substack{x \leqslant M+\delta \\ y \leqslant M+\delta}} \left(\left| \sum_{k=1}^{p} \varphi_k(x)\, \psi_k(y) \right| - B(1+x)(1+y) \right) \leqslant 1$$

for some $\delta > 0$. With a function g as before, we set

$$\psi^{(p)}(x, y) = \sum_{k=1}^{p} g(x)\, \varphi_k(x)\, h(y)\, \psi_k(y)$$

If $\pi_s^n = \frac{1}{2}(\alpha_s^n + u_s)$, we then have

$$\begin{aligned}
&\limsup |E[\langle (\alpha_s^n - u_s)^{\otimes 2}, \psi \rangle]| \\
&\quad \leqslant \limsup \left| E\left[\int_0^\infty \int_0^\infty \psi^{(p)}(x, y)(\alpha_s^n - u_s)(dx)(\alpha_s^n - u_s)(dy) \right] \right| \\
&\quad + 4 \sup_n E\left[\int_0^M \int_0^M \left| \psi(x, y) - \sum_{k=1}^{p} \varphi_k(x)\, \psi_k(y) \right| \pi_s^n(dx)\, \pi_s^n(dy) \right] \\
&\quad + 4 \sup_n E\left[\int_0^\infty \int_M^\infty (|\psi(x, y)| + |\psi^{(p)}(x, y)|) \pi_s^n(dx)\, \pi_s^n(dy) \right] \\
&\quad + 4 \sup_n E\left[\int_M^\infty \int_0^\infty (|\psi(x, y)| + |\psi^{(p)}(x, y)|)\, \pi_s^n(dx)\, \pi_s^n(dy) \right]
\end{aligned}$$

The first term on the right-hand side is zero by Cauchy–Schwarz and (2), the second term is less than $\varepsilon/4$ by (3), and the last two terms are less than C'/M by Theorem 2.1; the lemma is thus established.

We start now to study the fluctuations

$$\eta_t^n = \sqrt{n}(\alpha_t^n - u_t)$$

(See ref. 12, pp. 126–127, for some comments about these signed measures.)

Using Ito's formula, one can see that the following real processes are martingales for the filtration $\mathscr{G}_t^n = \sigma(\alpha_s^n: 0 \leqslant s \leqslant t)$, where $\varphi \in bB(\mathfrak{R}_+)$:

$$\langle M_t^n, \varphi \rangle = \langle \eta_t^n, \varphi \rangle - \langle \eta_0^n, \varphi \rangle - \int_0^t \langle \eta_s^n, A(\tfrac{1}{2}(\alpha_s^n + u_s))\varphi \rangle\, ds \tag{4}$$

$$S_t^n(\varphi) = \langle M_t^n, \varphi \rangle^2 - \int_0^t \langle \alpha_s^n \otimes \alpha_s^n, \tfrac{1}{2} A_2 \varphi \rangle\, ds \tag{5}$$

We shall see that under assumption **(c′)**, which we will soon enumerate, $(M_t^n)_{t\geqslant 0}$ is a strongly integrable vector martingale in H_1', for each $n\geqslant 2$; this result will be crucial in our study. But first we have to define H_1.

We observe that

$$(A_1\varphi)(x,y)=\left\{\int_x^y \varphi(u)\,du-\tfrac{1}{2}(y-x)[\varphi(x)+\varphi(y)]\right\}\operatorname{sign}(y-x)$$

$$=\left[\tfrac{1}{2}\int_x^y (y-z)(x-z)\,\varphi''(z)\,dz\right]\operatorname{sign}(y-x) \tag{6}$$

$$\frac{\partial}{\partial x}(A_1\varphi)(x,y)=\left[\frac{1}{2}\int_x^y (y-z)\,\varphi''(z)\,dz\right]\operatorname{sign}(y-x) \tag{7}$$

$$\frac{\partial^2}{\partial x^2}(A_1\varphi)(x,y)=-\frac{1}{2}|y-x|\,\varphi''(x) \tag{8}$$

and we then choose the following Hilbert spaces:

1. H is the completion of $\mathscr{D}=\mathscr{D}(\mathfrak{R}_+,\mathfrak{R})$ (the set of infinitely differentiable functions with compact support) for the scalar product:

$$\langle\varphi,\psi\rangle=\varphi(0)\,\psi(0)+\varphi'(0)\,\psi'(0)+\int\varphi''(x)\,\psi''(x)(1+x^2)\,dx$$

2. H_0 is the completion of $\mathscr{D}$ for the scalar product:

$$\langle\varphi,\psi\rangle_0=\varphi(0)\,\psi(0)+\varphi'(0)\,\psi'(0)+\int\varphi''(x)\,\psi''(x)\,dx$$

3. H_1 is the completion of $\mathscr{D}$ for the scalar product:

$$\langle\varphi,\psi\rangle_1=\varphi(0)\,\psi(0)+\varphi'(0)\,\varphi'(0)+\int\frac{\varphi''(x)\,\psi''(x)}{(1+x^2)}\,dx$$

We denote by B, B_0, and B_1 the respective unit balls, by N, N_0, and N_1 the respective norms, and by $\tilde{N}$, $\tilde{N}_0$, and $\tilde{N}_1$ the dual ones. Observe that, if $\int_{\mathfrak{R}_+} y\,\mu(dy)<\infty$, $A(\mu)$ is a continuous operator from H to H_0 and from H_0 to H_1. Our central limit result will be obtained under **(a)**, **(b)**, and the following assumptions:

- **(c′)** $\sup_n E[\langle\alpha_0^n, x^6\rangle]<\infty$
- **(d′)** $\langle u_0, x^6\rangle<\infty$
- **(e)** $\sup_n E[\tilde{N}^2(\eta_0^n)]<\infty$
- **(f)** $\sup_n E[|\langle\eta_0^n, x^4\rangle|]<\infty$

These six assumptions will be referred to as our "initial assumptions."

Remark. Assumptions **(a)**, **(b)**, **(c′)**, **(e)**, and **(f)** are satisfied, for instance, when the initial variables are i.i.d., for each $n \geqslant 2$, and when

$$\exists a, \forall n \geqslant 2, \qquad X^n_{i,0} \leqslant a$$

3. PRELIMINARY RESULTS

In this section we demonstrate two basic results. First, we prove that under our initial assumptions, η^n and M^n are, for each $n \geqslant 2$, H_0'-valued processes. Then, we show that a deterministic differential equation related to (1) has a unique H_0'-valued solution. To obtain the second result, we will first show that M^n_t is a strongly integrable H_1'-valued variable.

Proposition 3.1.

1. $\forall n$, η^n is an H_0'-valued process.
2. $\forall n$, $\forall t$, M^n_t is strongly integrable in H_1'.

Proof. 1. Writing $\varphi(x) = \varphi(0) + x\varphi'(0) + \int_0^x (x-z)\, \varphi''(z)\, dz$ and using Cauchy–Schwarz, for all $\varphi \in H_0$ we have

$$\begin{aligned}
|\langle \eta^n_t, \varphi \rangle| &= \left| \varphi'(0) \langle \eta^n_t, x \rangle + \left\langle \eta^n_t, \int_0^x (x-z)\, \varphi''(z)\, dz \right\rangle \right| \\
&\leqslant |\varphi'(0)|\, |\langle \eta^n_0, x \rangle| + \sqrt{n} \left\langle \alpha^n_t, \int_0^x (x-z)\, |\varphi''(z)|\, dz \right\rangle \\
&\quad + \sqrt{n} \left\langle u_t, \int_0^x (x-z)\, |\varphi''(z)|\, dz \right\rangle \\
&\leqslant \sqrt{n} \left[\left(\frac{1}{n} \sum_{i=1}^n X^n_i(0) + \int_0^\infty x u_0(dx) \right) |\varphi'(0)| \right. \\
&\quad \sqrt{n} \left\langle \alpha^n_t, \left(\int_0^x (x-z)^2\, dz \right)^{1/2} \left(\int_0^\infty |\varphi''(z)|^2\, dz \right)^{1/2} \right\rangle \\
&\quad \left. + \sqrt{n} \left\langle u_t, \left(\int_0^x (x-z)^2\, dz \right)^{1/2} \left(\int_0^\infty |\varphi''(z)|^2\, dz \right)^{1/2} \right\rangle \right] \\
&\leqslant \sqrt{n} \left[\frac{1}{n} \sum_{i=1}^n X^n_i(0) + \int_0^\infty x\, u_0(dx) \right. \\
&\quad \left. \times \frac{1}{\sqrt{3}} \left(\frac{1}{n} \sum_{i=1}^n X^n_i(t)^{3/2} \right) + \int_0^\infty \frac{x^{3/2}}{\sqrt{3}} u_t(dx) \right] N_0(\varphi)
\end{aligned}$$

Furthermore, $\langle \eta^n_t, \varphi \rangle$ is a real random variable for all $\varphi \in \mathscr{D}$. Since $\mathscr{D}$ generates the Borel σ-field of H_0', the conclusion follows.

2. For $y > x$ we observe that

$$
\begin{aligned}
|(\varepsilon^r \varphi)(x, y)| &= \left| \int_0^{(1-r)(y-x)} \int_{y-v}^{x+v} \varphi''(u)\, du\, dv \right| \\
&\leqslant \int_0^{(1-r)(y-x)} \int_{y-v}^{x+v} (1+u) \left| \frac{\varphi''(u)}{(1+u^2)^{1/2}} \right| du\, dv \\
&= \left[\int_0^{(1-r)(y-x)} \left(\int_{y-v}^{x+v} (1+u)^2\, du \right)^{1/2} dv \right] N_1(\varphi)
\end{aligned}
$$

The last inequality was obtained by applying Cauchy–Schwarz. Rough estimations give

$$
\begin{aligned}
\int_0^{(1-r)(y-x)} \left(\int_{y-v}^{x+v} (1+u)^2\, du \right)^{1/2} dv &\leqslant \int_0^{y} \left(\int_0^{2y} (1+u)^2\, du \right)^{1/2} dv \\
&\leqslant C_1(1 + y^{5/2})
\end{aligned}
$$

so if one takes a CONS $(\varphi_k)_{k \geqslant 1}$ in H_1, Parseval's identity gives

$$
\sum_{k \geqslant 1} ((\varepsilon^r \varphi_k)(x, y))^2 \leqslant C_2(1 + y^5)
$$

and

$$
\sum_{k \geqslant 1} \Lambda_2 \varphi_k(x, y) \leqslant C_3(1 + y^6)
$$

By symmetry we then have

$$
\sum_{k \geqslant 1} \Lambda_2 \varphi_k(x, y) \leqslant C_4(1 + x^6 + y^6)
$$

for all x, y. Therefore,

$$
\begin{aligned}
E[\tilde{N}_1^2(M_t^n)] &= \sum_{k \geqslant 1} E[\langle M_t^n, \varphi_k \rangle^2] \\
&= \int_0^t E\left[\left\langle \alpha_s^n \otimes \alpha_s^n, \sum_{k \geqslant 1} \tfrac{1}{2} \Lambda_2 \varphi_k \right\rangle \right] ds \\
&\leqslant C_5\, t \left(1 + \sup_{0 \leqslant s \leqslant t} E[\langle \alpha_s^n, x^6 \rangle] \right) < \infty
\end{aligned}
$$

The proposition is proved.

Part 2 of this proposition allows us to conclude that M^n is an H_1'-valued martingale. The next result is also crucial for what follows; it shows that a *deterministic* differential equation related to (1) has a unique solution in H_0'.

Proposition 3.2. Let:

1. $t \mapsto \pi_t$, in $D([0, T], \mathcal{M}_1^+(\Re_+))$, be such that $\int_0^\infty y^5 \pi_t(dy) < \infty$ for $0 \leqslant t \leqslant T$.
2. $A^*(\pi_t)$: $H_1' \to H_0'$ be defined by $\langle A^*(\pi_t)\eta, \varphi \rangle = \langle \eta, A(\pi_t)\varphi \rangle$, for $\varphi \in H_0$.
3. $t \mapsto M_t$ be a measurable mapping from $[0, T]$ to H_1' such that $\int_0^T \tilde{N}_1(M_t)\, dt < \infty$ and $M_0 = \eta_0$.

Then the equation

$$\eta_t = M_t + \int_0^t A^*(\pi_s)\eta_s\, ds$$

has a unique solution in H_0', given by

$$\eta_t = \eta_t(0)\delta_0 - \eta_t(1)\delta_0' + \eta_t(3)$$

where

$$\eta_t(0) = M_t(0)$$

$$\eta_t(1) = M_t(1)$$

$$\eta_t(3) = \eta_t''(2)$$

with

$$\eta_t(2; x) = M_t(2; x) + \left\{\left[\exp \int_0^t A^*(\pi_s)\, ds\right] \eta_0\right\}(2; x)$$
$$+ \int_0^t \left\{\left[\exp \int_s^t A^*(\pi_v)\, dv\right] A^*(\pi_s) M_s\right\}(2; x)\, ds$$

[Note: We identify H_0' with $\Re \otimes \Re \otimes L^2(dx)$.]

Proof. To prove uniqueness, we note that the difference $\hat{\eta}$ between two solutions has to satisfy

$$\hat{\eta}_t = \int_0^t A^*(\pi_s)\hat{\eta}\, ds$$

But, in general, we have

$$(A^*(\pi_s)\eta_s)(0)=0$$
$$(A^*(\pi_s)\eta_s)(1)=0$$
$$(A^*(\pi_s)\eta_s)(2;x)= -\tfrac{1}{2}\left\{[\eta_s(0)x+\eta_s(1)]\int_x^\infty (y-x)\,\pi_s(dy) + \eta_s(2;x)\int_0^\infty |y-x|\,\pi_s(dy)\right\} \tag{9}$$

It is then easy to see that $\hat{\eta}_t(0)=0$, $\hat{\eta}_t(1)=0$; from which we obtain

$$|\hat{\eta}_t(2;x)|\leqslant \frac{c+x}{2}\int_0^t |\hat{\eta}_s(2;x)|\,ds$$

Uniqueness then follows from Gronwall's lemma.

To show that the given η_t is a solution in H_0', we have to see that each of its terms is in H_0'. First, M_t has been chosen to be in H_1' so it is also in H_0'. Second, η_0 is also in H_1' and from

$$\begin{aligned}\left\{\left[\exp\int_0^t A^*(\pi_s)\,ds\right]\eta_0\right\}(2;x) &= \tfrac{1}{2}[\eta_0(0)x+\eta_0(1)] \\ &\quad\times\int_0^t\left\{\exp\left[-\tfrac{1}{2}\int_s^t\int_0^\infty |y-x|\,\pi_v(dy)\,dv\right]\right\}\int_x^\infty (y-x)\,\pi_s(dy)\,ds \\ &\quad+\left\{\exp\left[-\tfrac{1}{2}\int_0^t\int_0^\infty |y-x|\,\pi_v(dy)\,dv\right]\right\}\eta_0(2;x)\end{aligned} \tag{10}$$

one easily deduces, since $\int_0^\infty y^5\pi_s(dy)<\infty$, that the second term is in H_0'. Third, from (9) with M_s instead of η_s, we have that $A^*(\pi_s)M_s$ is in H_0'; this concludes the proof, since

$$\left\{\left[\exp\int_s^t A^*(\pi_v)\,dv\right]A^*(\pi_s)M_s\right\}(2;x) = \left\{\exp\left[-\tfrac{1}{2}\int_0^t\int_0^\infty |y-x|\,\pi_v(dy)\,dv\right]\right\}(A^*(\pi_s)M_s)(2;x)$$

Indeed, this last expression follows from (10) with $A^*(\pi_s)M_s$ replacing η_0.

4. RELATIVE COMPACTNESS

The next result implies that, for each $n \geqslant 2$, η^n and M^n have their paths in $D([0, T], H'_w)$, where H'_w is the weak dual of H. It is also a starting point for showing relative compactness because it makes possible, as in ref. 3, the use of a criterion in ref. 5.

Theorem 4.1. Under our initial assumptions, we have the following results:

1. $\sup_n E[\sup_{0 \leqslant t \leqslant T} \tilde{N}_1^2(M_t^n)] < \infty$.
2. $\sup_n E[\sup_{0 \leqslant t \leqslant T} \tilde{N}^2(\eta_t^n)] < \infty$.

Proof. 1. By part 2 of Proposition 3.1, we have already established that

$$E[\tilde{N}_1^2(M_T^n)[\leqslant C_5 T(1 + \sup_{0 \leqslant s \leqslant T} E[(X_{1,s}^n)^6])$$

But, noticing that $\tilde{N}(M_t^n) = \sup_{\varphi \in B_1} \langle M_t^n, \varphi \rangle$ is a positive submartingale, we only have to use Doob's inequality to obtain the desired result.

2. Proposition 3.2 allows us to write, for $\varphi \in H$ and $\pi_s^n = \frac{1}{2}(\alpha_s^n + u_s)$,

$$\langle \eta_t^n, \varphi \rangle = \langle M_t^n, \varphi \rangle + \langle \eta_0^n, S(0, t) \varphi \rangle + \int_0^t \langle M_s^n, A(\pi_s^n) S(s, t) \varphi \rangle \, ds$$

where

$$S(s, t) = \exp \int_s^t A(\pi_v^n) \, dv, \qquad s \leqslant t$$

By (6)–(8), we have that

$$(A(\pi_s^n) \varphi)(x) = (A(\pi_s^n) \varphi)(0) + (A(\pi_s^n) \varphi)'(0) x - \tfrac{1}{2} \int_0^x (x - z) \int_0^\infty |z - y| \, \pi_s^n(dy) \, \varphi''(z) \, dz$$

We then put

$$(B_s \varphi)(x) = (A(\pi_s^n) \varphi)(0) + (A(\pi_s^n) \varphi)'(0) x$$

$$(C_s \varphi)(x) = -\tfrac{1}{2} \int_0^x (x - z) \int_0^\infty |z - y| \, \pi_s^n(dy) \, \varphi''(z) \, dz$$

and observe that

$$(C_s + B_s) B_v = 0, \qquad C_s B_v = 0, \qquad B_s B_v = 0$$

since $A(\pi_s^n)(a+bx) \equiv 0$. We are then able to write (4) as the sum of five terms:

$$\langle \eta_t^n, \varphi \rangle = \langle M_t^n, \varphi \rangle + \left\langle \eta_0^n, \left[\left(\exp \int_0^t C_v \, dv \right) + \int_0^t B_v \, dv \sum_{k \geqslant 1} \left(\int_0^t C_v \, dv \right)^{k-1} \Big/ k! \right] \varphi \right\rangle + \int_0^t \left\langle M_s^n, (B_s + C_s) \left(\exp \int_s^t C_v \, dv \right) \varphi \right\rangle ds$$

The second result is then a consequence of the following five estimates.

1. Using the same lines of proof as those used in proving part 2 of Proposition 3.1, but for a CONS $(\varphi_k)_{k \geqslant 1}$ in H, we have, by Doob's inequality,

$$E[\sup_{0 \leqslant t \leqslant T} \sup_{\varphi \in B} \langle M_t^n, \varphi \rangle^2] \leqslant 4E[\sup_{\varphi \in B} \langle M_T^n, \varphi \rangle^2] \leqslant CT \sup_{0 \leqslant s \leqslant T} E[(X_{1,s}^n)^4]$$

2. $\sup_{0 \leqslant t \leqslant T} \sup_{\varphi \in B} \langle \eta_0^n, (\exp \int_0^t C_v \, dv) \varphi \rangle^2 \leqslant \tilde{N}^2(\eta_0^n)$.
3. The identities

$$(A(\mu)\varphi)'(0) = \tfrac{1}{2} \int_0^\infty [\varphi(y) - \varphi(0) - y\varphi'(0)] \, \mu(dy) = \tfrac{1}{2} \int_0^\infty \int_0^y (y-x) \, \varphi''(x) \, dx \, \mu(dy)$$

give

$$\left\langle \eta_0^n, \left[\int_0^t B_v \, dv \sum_{k \geqslant 1} \left(\int_0^t C_v \, dv \right)^{k-1} \Big/ k! \right] \varphi \right\rangle^2 = \int_0^t \int_0^\infty \int_0^y (y-x) \frac{1 - e^{-b_n(t,x)}}{b_n(t,x)} \varphi''(x) \, dx \, \pi_s^n(dy) \, ds \langle \eta_0^n, x \rangle$$

where

$$b_n(t,x) = \tfrac{1}{2} \int_0^t \int_0^\infty |x - z| \, \pi_s^n(dz) \, ds$$

This implies that

$$E\left[\sup_{0\leqslant t\leqslant T}\sup_{\varphi\in B}\left\langle \eta_0^n,\left[\int_0^t B_v\,dv\sum_{k\geqslant 1}\left(\int_0^t C_v\,dv\right)^{k-1}\Big/k!\right]\varphi\right\rangle^2\right]$$

$$\leqslant E\left[C\sup_{0\leqslant t\leqslant T} t\int_0^t\left(\int_0^\infty y^{3/2}\pi_s^n(dy)\right)^2 ds\ \langle\eta_0^n, x\rangle^2\right]$$

$$\leqslant CT^2\sup_{0\leqslant s\leqslant T} E\left[\left(\int_0^\infty y^{3/2}\pi_s^n(dy)\right)^2\langle\eta_0^n, x\rangle^2\right]$$

$$\leqslant CT^2\sup_{0\leqslant s\leqslant T}\left(E\left[\int_0^\infty y^6\pi_s^n(dy)\right]\right)^{1/2}(E[\langle\eta_0^n, x\rangle^4])^{1/2}$$

4. The fourth term is treated in the same way as the third, but with

$$\tfrac{1}{2}\int_0^t\int_0^\infty\int_0^y (y-x)\left\{\exp\left[-\tfrac{1}{2}\int_s^t |x-z|\,\pi_v^n(dz)\,dv\right]\right\}\varphi''(x)\,dx\,\pi_s^n(dy)\,ds$$

as the coefficient of $\langle\eta_0^n, x\rangle$ $(=\langle\eta_s^n, x\rangle=\langle M_s^n, x\rangle)$.

5. We let $L_{s,t}(\varphi)=C_s\,(\exp\int_s^t C_v\,dv)\varphi$ and we denote by B_0^c the ball with radius c in H_0. By the second initial assumption, we have that

$$|(L_{s,t})(\varphi)''(x)|\leqslant c(1+x^2)^{1/2}|\varphi''(x)|$$

which is to say

$$N_0(L_{s,t}(\varphi))\leqslant cN(\varphi)$$

We then obtain

$$E[\sup_{0\leqslant s\leqslant t\leqslant T}\sup_{\varphi\in B}\langle M_s^n, L_{s,t}(\varphi)\rangle^2]\leqslant E[\sup_{0\leqslant s\leqslant T}\sup_{\varphi\in B_0^c}\langle M_s^n,\varphi\rangle^2]$$

$$\leqslant 4E[\sup_{\varphi\in B_0^c}\langle M_T^n,\varphi\rangle^2]$$

$$\leqslant CT\sup_{0\leqslant s\leqslant T} E[(X_{1,s}^n)^4]$$

Under our initial assumptions, these five estimates yield the second result.

Theorem 4.2. Under our initial assumptions, the laws $\{\tilde{P}^n\}$ of the fluctuation processes are relatively compact for the weak convergence on $D([0, T], H'_w)$ and any limit law $\tilde{P}$ has its support in $C([0, T], H'_w)$.

Proof. The space H' endowed with its *weak* topology is a Lusin space. Moreover, since H'_w is the weak dual of a separable Fréchet space, the space $D([0, T], H'_w)$ (with the associated Skohorod topology) is also a Lusin space (Fernique,[5] Theorem 3.2.1). To show the relative compactness of the laws of the fluctuation processes $\{\eta^n\}_{n\geqslant 2}$, it is enough to verify (Fernique,[5] Theorem 4.4) the following two conditions:

(a) There exists a sequence $(K_m)_{m\geqslant 1}$ of weakly compact subsets of H'_w such that

$$\forall m \geqslant 1, \quad \forall n \geqslant 2, \qquad P^n\{\exists t \in [0, T] \mid \eta_t^n \notin K_m\} \leqslant 2^{-m}$$

(b) For all $\varphi \in \mathscr{D}$, the sequence of real processes $\{\langle \eta^n, \varphi\rangle\}_{n=2}^{\infty}$ has relatively compact laws in $\mathscr{M}_1^+(D([0, T], \mathfrak{R}))$.

Property (a) is straightforward. One just has to let

$$M = \sup_n E[\sup_{0\leqslant t\leqslant T} \tilde{N}^2(\eta_t^n)] < \infty$$

and to apply Tchebychev's inequality to the sets $K_m = \{\eta \in H'_w \mid \tilde{N}(\eta)^2 \leqslant M 2^m\}$. It remains to verify property (b). To establish this property, we will use the following lemmas, which will be proved under our initial assumptions.

Lemma 4.1. For any $\varphi \in \mathscr{D}$,

$$\lim_{M\uparrow\infty} \sup_n P^n\{\sup_{0\leqslant t\leqslant T} |\langle \eta_t^n, \varphi\rangle| > M\} = 0 \tag{11}$$

Lemma 4.1 is an easy consequence of Theorem 4.1. Now, in order to show the next lemma, we first introduce some notation and other tools which we will use in the sequel. For each function f in $D([0, T], \mathfrak{R})$ and $\delta > 0$ we set, as in Billingsley,[1]

$$W''(f, \delta) = \sup\{|f(t) - f(r)| \wedge |f(r) - f(s)|; 0 \leqslant s \leqslant r \leqslant t \leqslant T, t - s < \delta\}$$

We then have (Parthasarathy,[11] Chapter VII, Lemma 6.4, p. 235)

$$\sup_{\substack{s,t\in[0,T]\\ |t-s|<\delta}} |f(t) - f(s)| \leqslant 2W''(f, \delta) + \sup_{0\leqslant t\leqslant T} |f(t) - f(t-)| \tag{12}$$

We also denote $\tau_R^n = \inf\{t \geqslant 0\colon \tilde{N}(\eta_t^n)^2 > R\}$ and $Y_t^n = \langle \eta^n_{t\wedge\tau_R^n}, \varphi\rangle$. Hence, by Theorem 4.1, we have that

$$\lim_{R\uparrow\infty} \sup_n P^n\{\tau_R^n \leqslant T\} = 0 \tag{13}$$

Furthermore, since τ_R^n is a stopping time, the processes

$$\langle M^n_{t\wedge\tau_R^n}, \varphi\rangle = Y^n_t - \langle \eta^n_0, \varphi\rangle - \int_0^{t\wedge\tau_R^n} \langle \eta^n_s, A(\tfrac{1}{2}(\alpha^n_s + u_s))\varphi\rangle\, ds$$

$$S^n_{t\wedge\tau_R^n}(\varphi) = \langle M^n_{t\wedge\tau_R^n}, \varphi\rangle^2 - \int_0^{t\wedge\tau_R^n} \langle \alpha^n_s\otimes\alpha^n_s, \tfrac{1}{2}A_2\varphi\rangle\, ds$$

are martingales for the filtration $(\mathscr{G}^n_t)$.

We now state Lemma 4.2.

Lemma 4.2. Using the above notation, we have

$$E[\langle M^n_{t\wedge\tau_R^n}, \varphi\rangle^2 - \langle M^n_{r\wedge\tau_R^n}, \varphi\rangle^2 \mid \mathscr{G}^n_r] \leqslant C(t-r), \quad \text{a.s.}$$

Proof of Lemma 4.2. From the definition of $\langle M^n_{t\wedge\tau_R^n}, \varphi\rangle$, we have

$$\begin{aligned} &E\{\langle M^n_{t\wedge\tau_R^n}, \varphi\rangle^2 - \langle M^n_{r\wedge\tau_R^n}, \varphi\rangle^2 \mid \mathscr{G}^n_r\} \\ &= E\{S^n_{t\wedge\tau_R^n}(\varphi) - S^n_{r\wedge\tau_R^n}(\varphi) \mid \mathscr{G}^n_r\} + E\left\{\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n} \langle \alpha^n_u\otimes\alpha^n_u, \tfrac{1}{2}A_2\varphi\rangle\, du \mid \mathscr{G}^n_r\right\}\end{aligned}$$

Since the process $(S^n_{t\wedge\tau_R^n}(\varphi))$ is a martingale with respect to the filtration $(\mathscr{G}^n_t)$, the right-hand side of this equality may be written

$$E\left\{\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n}\left[\int_{\mathfrak{R}_+\times\mathfrak{R}_+} \tfrac{1}{2}(A_2)(x, y)\, \alpha^n_u(dx)\otimes\alpha^n_u(dy)\right] du \mid \mathscr{G}^n_r\right\}$$

This expression is bounded by

$$\begin{aligned} &E\left\{\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n}\left[\int_{\mathfrak{R}_+\times\mathfrak{R}_+} C_1(x+y)\, \alpha^n_u(dx)\otimes\alpha^n_u(dy)\right] du \mid \mathscr{G}^n_r\right\} \\ &\quad = 2C_1 E\left\{\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n} \langle \alpha^n_u, x\rangle\, du \mid \mathscr{G}^n_r\right\} \\ &\quad = 2C_1 E\left\{\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n} \frac{1}{n}\sum_{i=1}^n X^n_{i,u}\, du \mid \mathscr{G}^n_r\right\}\end{aligned}$$

Finally, the energy being conserved, we have that

$$\begin{aligned} &E\{\langle M^n_{t\wedge\tau_R^n}, \varphi\rangle^2 - \langle M^n_{r\wedge\tau_R^n}, \varphi\rangle^2 \mid \mathscr{G}_r\} \\ &\quad \leqslant 2C_1 E\left\{(t-r)\wedge\tau_R^n \cdot \frac{1}{n}\sum_{i=1}^n X^n_{i,0}\, du \mid \mathscr{G}^n_r\right\} \\ &\quad \leqslant 2C_1(t-r)\, E\left\{\frac{1}{n}\sum_{i=1}^n X^n_{i,0}\, du \mid \mathscr{G}^n_r\right\}\end{aligned}$$

Hence, using Condition 3(**b**) of Theorem 2.1, we obtain

$$E\{\langle M^n_{t\wedge\tau^n_R},\varphi\rangle^2-\langle M^n_{r\wedge\tau^n_R},\varphi\rangle^2\,|\,\mathscr{G}^n_r\}\leqslant C(t-r),\quad \text{a.s.}$$

Lemma 4.3. Under the hypothesis of Theorem 2.1 and with the above notation, we have

$$E\left\{\left|\int_{r\wedge\tau^n_R}^{t\wedge\tau^n_R}\langle\eta^n_s,A(\tfrac{1}{2}(\alpha^n_s+u_s))\varphi\rangle\,ds\right|^2\middle|\mathscr{G}^n_r\right\}\leqslant K_T(\varphi)(t-r),\qquad \text{a.s.}$$

where $K_T(\varphi)$ is a constant depending only on T and φ.

Proof of Lemma 4.3. Let $\pi^n_s=\frac{1}{2}(\alpha^n_s+u_s)$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned}
&E\left\{\left|\int_{r\wedge\tau^n_R}^{t\wedge\tau^n_R}\langle\eta^n_s,A(\pi^n_s)\varphi\rangle\,ds\right|^2\middle|\mathscr{G}^n_r\right\}\\
&\quad\leqslant E\left\{(t-r)\wedge\tau^n_R\int_{r\wedge\tau^n_R}^{t\wedge\tau^n_R}\langle\eta^n_s,A(\pi^n_s)\varphi\rangle^2\,ds\,|\,\mathscr{G}^n_r\right\}\\
&\quad\leqslant(t-r)\,E\left\{\int_{r\wedge\tau^n_R}^{t\wedge\tau^n_R}\langle\eta^n_s,A(\pi^n_s)\varphi\rangle^2\,ds\,|\,\mathscr{G}^n_r\right\}\\
&\quad\leqslant(t-r)\,E\left\{\int_{r\wedge\tau^n_R}^{t\wedge\tau^n_R}N^2(\eta^n_s)\,N^2(A(\pi^n_s)\varphi)\,ds\,|\,\mathscr{G}^n_r\right\}
\end{aligned}\tag{14}$$

Writing $[0,K]$ for the support of φ and C for the different positive constants, we now show that $N^2(A(\pi^n_v)\varphi)\leqslant C(1+K^2)\,N^2(\varphi)$, a.s.

Observe that

$$\begin{aligned}
N^2(A(\pi^n_v)\varphi)&\leqslant C\int[(A(\pi^n_v)\varphi)''(x)]^2(1+x^2)\,dx\\
&\leqslant C\int_{[0,K]}\left\{\left[-\tfrac{1}{2}\int_{\Re_+}|x-y|\,\pi^n_v(dy)\right]\varphi''(x)\right\}^2(1+x^2)\,dx
\end{aligned}$$

Since for any $v\in[0,T]$ the integral $\int y\,u_v(dy)$ is finite and according to Condition 3(**b**) of Theorem 2.1, there exists $a>0$ such that $\forall n$, $\sum_{i=1}^n X^n_{i,0}/n\leqslant a$ a.s., then for each $v\in[0,T]$, the integral $\int_{\Re_+}y\,\pi^n_v(dy)$ is finite a.s. and we have that

$$N^2(A(\pi^n_v)\varphi)\leqslant C(1+K^2)\,N^2(\varphi),\quad \text{a.s.}$$

Therefore, the inequality (14) can be written

$$E\left\{\left|\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n}\langle\eta_s^n, A(\pi_s^n)\varphi\rangle\, ds\right|^2\Big|\mathcal{G}_r^n\right\}$$
$$\leqslant C(1+K^2)\,N^2(\varphi)(t-r)\,E\left\{\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n} N^2(\eta_v^n)\,dv\,|\,\mathcal{G}_r^n\right\},\quad \text{a.s.}$$

But

$$E\left\{\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n} N^2(\eta_v^n)\,dv\,|\,\mathcal{G}_r^n\right\}\leqslant T\cdot R$$

so that by substitution we obtain that

$$E\left\{\left|\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n}\langle\eta_s^n, A(\pi_v^n)\varphi\rangle\, ds\right|^2\Big|\mathcal{G}_r^n\right\}=K_T(\varphi)(t-r),\quad \text{a.s.}$$

where $K_T(\varphi)$ depends only on T and φ. This is the desired result.

Lemma 4.4. For any triple $s\leqslant r\leqslant t$, we have:

1. $E\{|Y_t^n - Y_r^n|^2\,|\,\mathcal{G}_r^n\}\leqslant K_T(\varphi)(t-r)$, a.s.
2. $E\{|Y_t^n - Y_r^n|^2\,|Y_r^n - Y_s^n|^2\}\leqslant K_T'(\varphi)(t-r)^2$, a.s.

where $K_T(\varphi)$ and $K_T'(\varphi)$ are positive constants depending only on T and φ.

Proof of Lemma 4.4. 1. For any couple $r<t$, we have that

$$E\{|Y_t^n - Y_r^n|^2\,|\,\mathcal{G}_r^n\}\leqslant E\{\langle M^n_{t\wedge\tau_R^n},\varphi\rangle^2-\langle M^n_{r\wedge\tau_R^n},\varphi\rangle^2\,|\,\mathcal{G}_r^n\}$$
$$+2E\left\{\int_{r\wedge\tau_R^n}^{t\wedge\tau_R^n}\langle\eta_v^n, A(\pi_v^n)\rangle^2\,dv\,|\,\mathcal{G}_r^n\right\}$$

so that by Lemmas 4.2 and 4.3 we obtain

$$E\{|Y_t^n - Y_r^n|^2\,|\,\mathcal{G}_r^n\}\leqslant K_T(\varphi)(t-r),\quad \text{a.s.}$$

2. For any triple $s\leqslant r\leqslant t$, we have

$$E[(Y_t^n - Y_r^n)^2(Y_r^n - Y_s^n)^2]=E[(Y_r^n - Y_s^n)^2\,E[(Y_t^n - Y_r^n)^2\,|\,\mathcal{G}_r^n]]$$
$$\leqslant K_T^2(\varphi)(t-r)(r-s)$$

Hence

$$E[|Y_t^n - Y_r^n|^2\,|Y_r^n - Y_s^n|^2]\leqslant K_T'(\varphi)(t-s)^2$$

Lemma 4.5. For all $\varepsilon > 0$, there exists an integer $N_0 \geqslant 1$ such that

$$\lim_{\delta \downarrow 0} \sup_{n \geqslant N_0} P^n[W''(Y^n, \delta) > \varepsilon/4] = 0$$

Lemma 4.5 follows immediately[1] from part 2 of Lemma 4.4.

We now give the proof of **(b)**. According to Theorem 15.5 of Billingsley,[1] it suffices to check the following two conditions:

(b$_1$) For all $\varphi \in \mathscr{D}$

$$\lim_{M \uparrow \infty} \sup_n P^n[\sup_{0 \leqslant t \leqslant T} |\langle \eta_t^n, \varphi \rangle| > M] = 0$$

(b$_2$) For any $\varepsilon > 0$ and $\varphi \in \mathscr{D}$, there exists $\delta > 0$ and an integer $N_0 \geqslant 2$ such that

$$\sup_{n \geqslant N_0} P^n[\sup_{\substack{s,t \in [0,T] \\ |t-s| < \delta}} |\langle \eta_t^n, \varphi \rangle - \langle \eta_s^n, \varphi \rangle| \geqslant \varepsilon] \leqslant \varepsilon$$

Property **(b$_1$)** is exactly Lemma 4.1 as stated above. We therefore verify **(b$_2$)**.

Let us fix $\varepsilon > 0$ and $\varphi \in \mathscr{D}$ and afterward choose an integer N_0 large enough that when $n \geqslant N_0$,

$$\frac{4}{\sqrt{n}} \|\varphi\|_\infty \leqslant \frac{\varepsilon}{2}$$

Since the probability that more than two components of $X^n(t)$ change at the same time is zero, we then have

$$P^n\left[\sup_{0 \leqslant t \leqslant T} |\langle \eta_t^n, \varphi \rangle - \langle \eta_{t-}^n, \varphi \rangle| \leqslant \frac{4}{\sqrt{n}} \|\varphi\|_\infty\right] = 1$$

which implies that

$$P^n\left[\sup_{0 \leqslant t \leqslant T} |\langle \eta_t^n, \varphi \rangle - \langle \eta_{t-}^n, \varphi \rangle| \leqslant \frac{\varepsilon}{2}\right] = 1$$

On the other hand, because of property (13), Lemma 4.5 implies

$$\lim_{\delta \downarrow 0} \sup_{n \geqslant N_0} P^n[W''(\langle \eta^n, \varphi \rangle, \delta) > \varepsilon/4] = 0$$

Hence, by inequality (12), we get

$$\lim_{\delta \downarrow 0} \sup_{n \geqslant N_0} P^n[\sup_{\substack{s,t \in [0,T] \\ |t-s| < \delta}} |\langle \eta_t^n, \varphi \rangle - \langle \eta_s^n, \varphi \rangle| \geqslant \varepsilon] = 0$$

This is precisely property **(b)**, so that the proof of Theorem 4.2 is complete.

5. CONVERGENCE

Adding to our initial assumptions the hypothesis that $\{\eta_0^n\}_{n \geq 2}$ converges to η_0 weakly in H'_w, we prove that the sequence $\{\eta^n\}_{n \geq 2}$ on $D([0, T], H'_w)$ has a unique limit point; hence, this sequence converges. To do so, we will first show that, for each limit point, some expressions are martingales. This will allow us to identify a Wiener process on $C([0, T], H'_w)$; this process has nonstationary increments. Finally, we shall see that any limit point is a solution of a Langevin equation; uniqueness then follows from Proposition 3.2.

Theorem 5.1. Under all our initial assumptions, we have that the fluctuation process laws $\tilde{P}^n$ on $D([0, T], H'_w)$ converge to a continuous process which is the unique solution of the Langevin equation:

$$\eta_t = \eta_0 + \int_0^t A^*(u_s)\eta_s \, ds + W_t$$

Here W is a Wiener process whose quadratic characteristic is

$$\int_0^t \langle u_s \otimes u_s, \tfrac{1}{2} A_2 \varphi \rangle \, ds \tag{15}$$

Proof. By Proposition 3.2 and part 1 of Theorem 4.1, η can be written explicitly in terms of η_0 and W. To show the convergence of the fluctuation process laws, it is then enough to check[7] that:

(A) For each limit law $\tilde{P}$, W is a continuous martingale with non-random characteristic.

To establish **(A)**, we will use two propositions. In order to state the first of these, we introduce some notation.

We know that under P^n, for each $f \in C_b^3$ and $\varphi \in \mathscr{D}$,

$$f(\langle \eta_t^n, \varphi \rangle) - \int_0^t C^n(f, \varphi, \eta_s^n) \, ds$$

is a martingale, where

$$\begin{aligned} C^n(f, \varphi, \eta_s^n) = {} & \left\langle \eta_s^n, A\left(\frac{1}{2}(\alpha_s^n + u_s)\right)\varphi \right\rangle f'(\langle \eta_s^n, \varphi \rangle) \\ & + \frac{1}{2}\left\langle \alpha_s^n \otimes \alpha_s^n, \frac{1}{2} A_2 \varphi \right\rangle f''(\langle \eta_s^n, \varphi \rangle) \\ & + \frac{1}{12\sqrt{n}} a_n(s, \omega) \end{aligned}$$

with $a_n(s, \omega)$ such that

$$|a_n(s, \omega)| \leqslant \langle \alpha_s^n \otimes \alpha_s^n, |A_3| \varphi \rangle \|f'''\|_\infty$$

and

$$(|A_3| \varphi)(x, y) = \int_0^1 |\varphi \oplus \varphi(x^*(r), y^*(r)) - \varphi \oplus \varphi(x, y)|^3 r |x-y| \, dr$$

We then set

$$C(f, \varphi, \eta_s^n) = \langle \eta_s^n, A(u_s)\varphi \rangle f'(\langle \eta_s^n, \varphi \rangle) + \tfrac{1}{2} \langle u_s \otimes u_s, \tfrac{1}{2} A_2 \varphi \rangle f''(\langle \eta_s^n, \varphi \rangle)$$

and observe that

$$C^n(f, \varphi, \eta_s^n) - C(f, \varphi, \eta_s^n) = \frac{1}{2\sqrt{n}} \langle \eta_s^n \otimes \eta_s^n, A_1 \varphi \rangle f'(\langle \eta_s^n, \varphi \rangle) + \frac{1}{4} \langle \alpha_s^n \otimes \alpha_s^n - u_s \otimes u_s, A_2 \varphi \rangle f''(\langle \eta_s^n, \varphi \rangle) + \frac{1}{12\sqrt{n}} a_n(s, \omega)$$

We now state Proposition 5.1.

Proposition 5.1. Under our initial assumptions, for each $f \in C_b^3$ and each $\varphi \in \mathscr{D}$,

$$\lim_{n \to \infty} E[|C^n(f, \varphi, \eta_s^n) - C(f, \varphi, \eta_s^n)|] = 0$$

Proof. From $|A_3| \varphi(x, y) \leqslant C(x+y)$ and our second initial assumption, we see that the expectation of the last term goes to zero as n goes to infinity. Because of part 2 of Theorem 4.1 and our fourth initial assumption, the expectation of the first term is also going to zero. Indeed, remembering (6)–(8), we have

$$\langle \eta_s^n \otimes \eta_s^n, A_1 \varphi \rangle = \int_0^\infty \int_0^\infty \int_x^y \tfrac{1}{2}(z-y)(z-x) \varphi''(z) \, dz \, \mathrm{sign}(y-x) \, \eta_s^n(dx) \, \eta_s^n(dy) = \int_0^\infty \varphi''(z) \left[\int_z^\infty \int_0^z (z-y)(z-x) \, \eta_s^n(dx) \, \eta_s^n(dy) \right] dz$$

$$= \int_0^\infty \varphi''(z)\left[\int_0^\infty \int_0^z (z-y)(z-x)\,\eta_s^n(dx)\,\eta_s^n(dy) - \int_0^z \int_0^z (z-y)(z-x)\,\eta_s^n(dx)\,\eta_s^n(dy)\right] dz$$

$$= \int_0^\infty \varphi''(z)\left\{\left[-\int_0^\infty y\,\eta_0^n(dy)\right]\left[\int_0^z (z-x)\,\eta_s^n(dx)\right] - \left[\int_0^z (z-x)\,\eta_s^n(dx)\right]^2\right\} dz$$

and therefore, denoting by K the compact support of φ, we obtain

$$\begin{aligned} &E[|\langle \eta_s^n \otimes \eta_s^n, A_1 \varphi\rangle|] \\ &\leqslant C_1 \left\{E\left[|\langle \eta_0^n, y\rangle| \int_K \left|\int_0^z (z-x)\,\eta_s^n(dx)\right| dz\right] + E\left[\int_K \left(\int_0^z (z-x)\,\eta_s^n(dx)\right)^2 dz\right]\right\} \\ &\leqslant C_2 \left\{E[|\langle \eta_0^n, y\rangle|^2]^{1/2} E\left[\int_K \left(\int_0^z (z-x)\,\eta_s^n(dx)\right)^2 dz\right]^{1/2} + E\left[\int_K \left(\int_0^z (z-x)\,\eta_s^n(dx)\right)^2 dz\right]\right\} \\ &\leqslant C_3 \{E[|\langle \eta_0^n, y\rangle|^2]^{1/2}\, E[\tilde{N}^2(\eta_s^n)]^{1/2} + E[\tilde{N}^2(\eta_s^n)]\} \end{aligned}$$

The second term also goes to zero because of the following lemma.

Lemma 5.1. Under our initial assumptions,

$$\lim_n E[|\langle \alpha_s^n \otimes \alpha_s^n, \psi\rangle - \langle u_s \otimes u_s, \psi\rangle|] = 0$$

for all continuous symmetric $\psi: \mathfrak{R}_+ \times \mathfrak{R}_+ \to \mathfrak{R}$ such that

$$|\psi(x, y)| \leqslant B(1+x)(1+y)$$

Proof. Since

$$\langle \alpha_t^n \otimes \alpha_t^n - u_t \otimes u_t, \psi\rangle = \langle (\alpha_t^n - u_t)^{\otimes 2}, \psi\rangle + 2\langle (\alpha_t^n - u_t) \otimes u_t, \psi\rangle$$

the result follows from Lemma 2.1.

Proposition 5.2. Under our initial assumptions, every limit point of the sequence $\{\tilde{P}^n\}_{n\geqslant 2}$ is such that, for each $f\in C_b^3$ and each $\varphi\in\mathscr{D}$,

$$f(\langle\eta_t,\varphi\rangle)-\int_0^t C(f,\varphi,\eta_s)\,ds$$

is a martingale.

Proof. By Proposition 5.1 we just have to show, as in ref. 2, that for each bounded, continuous Φ,

$$\lim_{n'\to\infty}\tilde{E}^{n'}\left[\int_0^t\langle\eta_s,A(u_s)\varphi\rangle f'(\langle\eta_s,\varphi\rangle)\,\Phi(\eta)\,ds\right]$$
$$=\tilde{E}\left[\int_0^t\langle\eta_s,A(u_s)\varphi\rangle f'(\langle\eta_s,\varphi\rangle)\,\Phi(\eta)\,ds\right]\tag{16}$$

where $\{n'\}$ is a subsequence such that $\{\tilde{P}^{n'}\}$ weakly converges to a limit law $\tilde{P}$ and $\tilde{E}^{n'}$, $\tilde{E}$ are the corresponding expectations.

To establish (16), we prove two lemmas. In order to state these lemmas, we define the variable $g\colon D([0,T],H'_w)\to\mathfrak{R}$ by

$$g(\eta)=\int_0^t\langle\eta_s,A(u_s)\varphi\rangle f'(\langle\eta_s,\varphi\rangle)\,\Phi(\eta)\,ds$$

We also observe that, since a compact K of $D([0,T],H'_w)$ is metrizable, g is continuous on K if and only if g is sequentially continuous on K.

Lemma 5.2:

$$\lim_{M\to\infty}\int_{\{|g(\eta)|>M\}}|g(\eta)|\,\tilde{P}^n(d\eta)=0$$

Proof. We have that

$$\int_{\{|g(\eta)|>M\}}|g(\eta)|\,\tilde{P}^n(d\eta)\leqslant\frac{C}{M}\tilde{E}^n\,[\sup_{0\leqslant s\leqslant t}\tilde{N}(\eta_s)^2]$$

Hence, the lemma follows from part 2 of Theorem 4.1.

Lemma 5.3. The $\tilde{P}^{n'}\circ g^{-1}$ weakly converge to $\tilde{P}\circ g^{-1}$.

Proof. On a compact K we have

$$\sup_{\eta\in K}\sup_{0\leqslant s\leqslant t}\tilde{N}(\eta_s)<\infty$$

Therefore, g is sequentially continuous on K. On the other hand, since by Theorem 4.2 the sequence $\{\tilde{P}^{n'}\}$ is tight, there exists, for each $\varepsilon > 0$, a compact K_ε in $D([0, T], H'_w)$ such that

$$\inf_{n'} P^{(n')}\{\eta^{n'} \in K_\varepsilon\} > 1 - \varepsilon \qquad \text{and} \qquad \tilde{P}\{\eta \in K_\varepsilon\} > 1 - \varepsilon$$

Now, let k be a bounded (say by 1) continuous function, and set $h = k \circ g$. Now, since by Theorem 4.1, $\{\eta^{n'}\}$ is tight. We know that $h|_{K_\varepsilon}$ is continuous and thus may be extended to a continuous function $\tilde{h}$ still bounded by one; this can be done because $D([0, T], H'_w)$ is a regular Lusin space; thus it is normal. Therefore,

$$\begin{aligned}\limsup |\tilde{E}^{n'}[h] - \tilde{E}[h]| &\leqslant \limsup |\tilde{E}^{n'}[\tilde{h}] - \tilde{E}[\tilde{h}]| \\ &\quad + \limsup |\tilde{E}^{n'}[h - \tilde{h}]| \\ &\quad + \limsup |\tilde{E}[h - \tilde{h}]| \\ &\leqslant 0 + 2\varepsilon + 2\varepsilon\end{aligned}$$

Lemma 5.3 is thus proved, as is Proposition 5.2.

The proof of Theorem 5.1 is almost complete. Indeed, for a fixed $\varphi \in \mathscr{D}$, $\langle \eta^n, \varphi \rangle$ is bounded. Therefore, taking f such that $f(x) = x$ on a large interval, we define

$$\langle W_t, \varphi \rangle = \langle \eta_t, \varphi \rangle - \langle \eta_0, \varphi \rangle - \int_0^t \langle A^*(u_s)\eta_s, \varphi \rangle \, ds$$

Recalling (5) and our deductions, we see that W is a continuous martingale with nonrandom characteristic given by (15).

We conclude this work by pointing out that η is Gaussian as soon as (η_0, W) is.

REFERENCES

1. P. Billingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).
2. R. Ferland, Equations de Boltzmann scalaires: convergence, fluctuations et propagation du choas trajectorielle, Ph.D. thesis, Université de Sherbrooke (1989).
3. R. Ferland, X. Fernique, and G. Giroux, Compactness of the fluctuations associated with some generalized nonlinear Boltzmann equations; *Can. J. Math.* **44**:1192–1205 (1992).
4. R. Ferland and J. C. Roberge, Binomial Boltzmann Processes: Convergence of the Fluctuations, Technical Report 193, Département de mathématiques et d'informatique, Université du Québec à Montréal (1992).
5. X. Fernique, Convergence en loi des fonctions aléatoires continues ou cadlag, propriétés de compacité des lois, in *Lecture Notes in Mathematixcs*, No. 1485 (Springer-Verlag, Berlin, 1991), pp. 178–195.

6. K. Ito, *Foundation of Stochastic Differential Equations in Infinite Dimensional Spaces* (Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1984).
7. J. Jacob and A. N. Shiryaev, *Limit Theorems for Stochastic Processes* (Springer-Verlag, Berlin, 1987).
8. M. Kac, Some probabilistic aspects of the Boltzmann equation, *Acta Phys. Aust. Suppl.* **10**:379–400 (1973).
9. J. Keizer, *Statistical Thermodynamics of Nonequilibrium Processes* (Springer-Verlag, New York, 1987).
10. H. P. McKean, Fluctuation in the kinetic theory of gases, *Commun. Pure Appl. Math.* **28**:435–455 (1975).
11. K. R. Parthasarathy, *Probability Measures on Metric Spaces* (Academic Press, New York, 1969).
12. A. V. Skorohod, *Stochastic Equations for Complex Systems* (Reidel, Dordrecht, 1988).
13. H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer-Verlag, New York, 1991).
14. A.-S. Sznitman, Équations de type de Boltzmann, spatialement homogènes, *Z. Wahrsch. Verw. Gebiete* **66**:559–592 (1984).
15. H. Tanaka, Fluctuation theory for Kac's one-dimensional model of Maxwellian molecules, *Sankhyā Ser. A* **44**:23–26 (1982).
16. K. Uchiyama, Fluctuations of Markovian systems in Kac's caricature of a Maxwellian gas, *J. Math. Soc. Jpn.* **35**(3):477–499 (1983).
17. K. Uchiyama, A fluctuations problem associated with the Boltzmann equation for a gas a molecules with a cutoff potential, *Jpn. J. Math.* **9**:27–53 (1983).
18. W. Wagner, A convergence proof for Bird's direct simulation Monte Carlo method for the Boltzmann equation, *J. Stat. Phys.* **66**:689–722 (1992).